This course is the beginning of multivariable calculus, which is calculus that involves many variables.

Calculus so far (Calculus I or AP Calculus):

derivative/integral of a function, that spits out one value when you put one number in.

Examples: $f(x) = \cos x$

$f(x) = x^2$

$f(x) =$ the average temperature of NYC at a given time x.

Calculus is useful for dealing with these functions, but the major problem is that the real world often cannot be modeled by such single-valued single-variable functions.

Examples: · The world is in 3D, so needs 3 numbers to describe a location (3-variable)

· Let's say you want to describe the relationship between the prices of two stocks, then at any given time, you have to deal with 2 values (the two prices)

Therefore, learning multivariable calculus will give you tools to analyze these functions that model real-world phenomena better.

Some examples

- Given a flight trajectory, we can compute the amount of jet fuel needed to complete the flight.
- One can make a wind forecast given the current map of atmospheric pressure.
- Multivariable calculus is the mathematical background needed for the basics of machine learning, for example support vector machine.
- If you run a chocolate factory, you may want to optimize the efficiency of the factory by using calculus. For example, from your experience, you have 2 functions, one about the total cost of production, one about the total amount produced, and both depend on two variables, labor (the amount of manpower) and capital (like the amount of chocolate-making machines). Given the restrictions on the capital & labor, one can find the optimal situation where the profit is maximized.
- There are many, many partial differential equations (equations involving differentiation with respect to more than one variable) that describe various aspects of the world, like Schrödinger equation, heat equation, Shephard's lemma, Black-Scholes equation, reaction-diffusion equation, Fick's law, Burgers' equation...

We begin with (a review of) coordinate systems, several ways to express the location of a point in terms of numbers.

There are more than one because sometimes it's more convenient to use one over the others (it's true, I promise.).

Coordinate systems in a plane (2D):

- Rectangular coordinates = Cartesian coordinates
- Polar coordinates

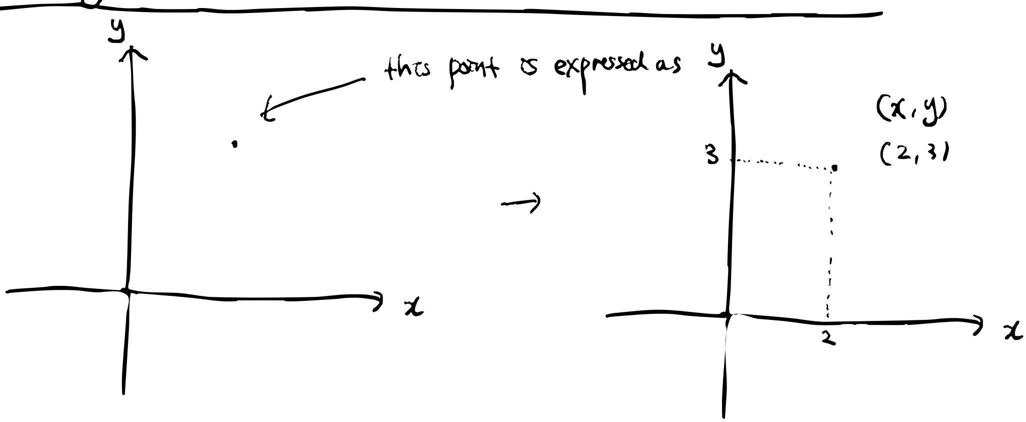
related

Coordinate systems in a space (3D):

- Rectangular coordinates = Cartesian coordinates
- Cylindrical coordinates
- Spherical coordinates

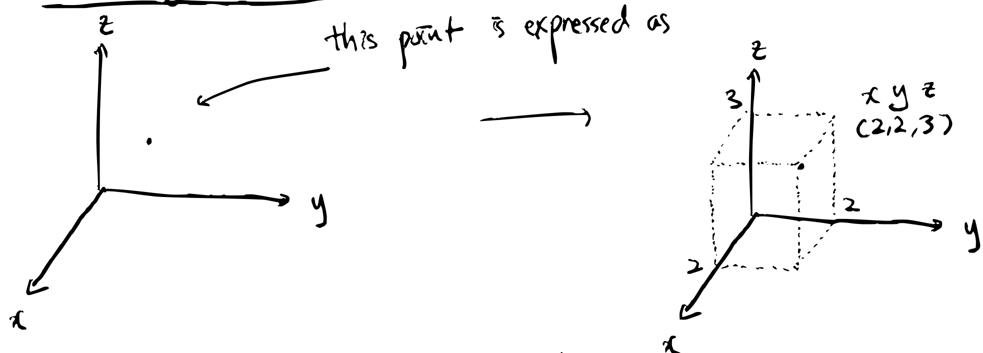
related

Rectangular coordinates = Cartesian coordinates in 2D



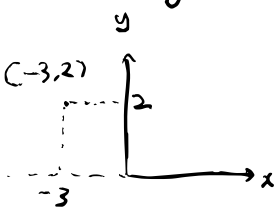
(2, 3) means 2 towards the positive x-direction (horizontally to the right)
3 towards the positive y-direction (vertically to the top)

Rectangular coordinates = Cartesian coordinates in 3D.



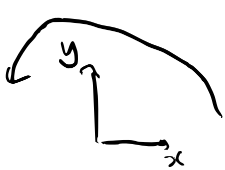
(2,2,3) means 2 towards the positive x-direction
2 towards the positive y-direction
3 towards the positive z-direction

* negative numbers can appear, if the point is located on the opposite direction (negative direction)

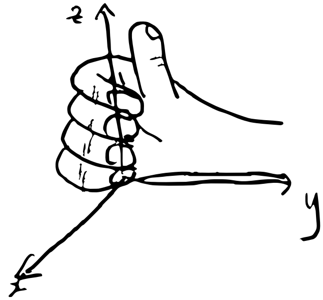


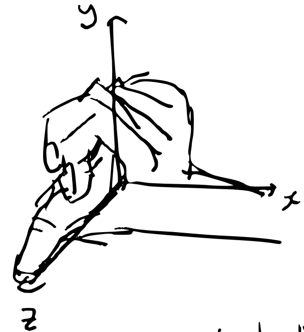
* the position of the axes determine the coordinate, and they must be positioned so that

· In 2D, x comes before y when you rotate counter-clockwise



· In 3D, the direction of z follows the right hand rule, namely when you position your right hand with the thumb up, then your thumb points the (positive) z-direction, and your figers curl from the positive z-direction to the positive y-direction.

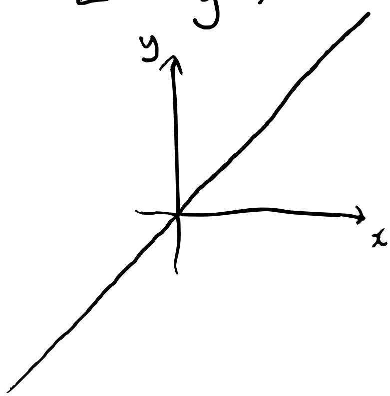




(sorry about the picture)

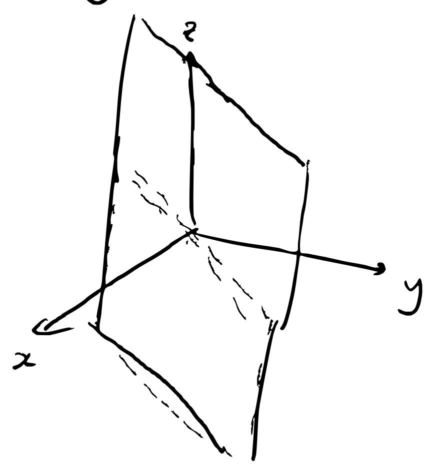
Rectangular coordinates are great in expressing straight shapes like lines & planes.

Ex: $y = x$ (2D)



(extends indefinitely)

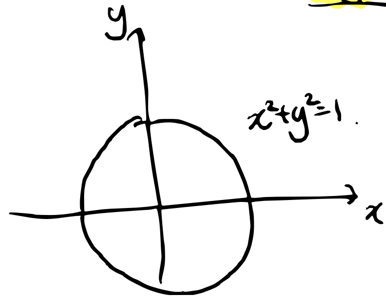
$y = x$ (3D)



(extends indefinitely)

A circular shape can also be expressed quite nicely.

For example, a circle centered at the origin (point $(0,0)$) with radius 1 has equation $x^2+y^2=1$.



(This is because the distance between the origin and the point (x,y) is $\sqrt{x^2+y^2}$, by Pythagorean rule.)

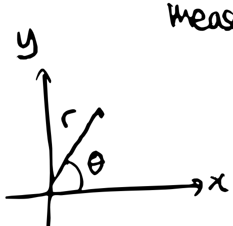
Polar coordinates (2D)

There is a coordinate system that suits the best in expressing circles around the origin.

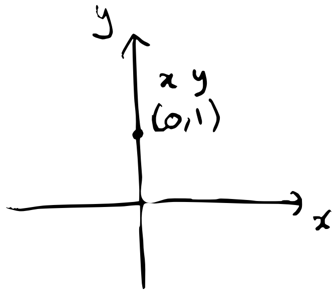
It uses two measurements different from x and y:

r: the distance from the origin

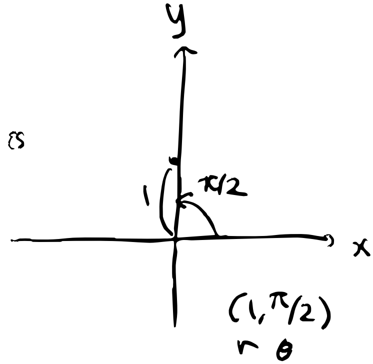
θ: the angle the line from the origin to the point makes with the positive x-axis, measured counterclockwise, in radians (a whole rotation takes 2π radians)

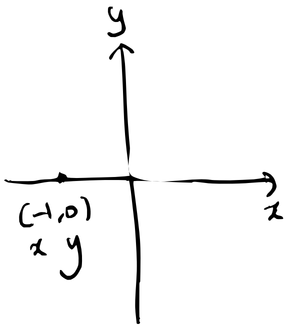


Ex:

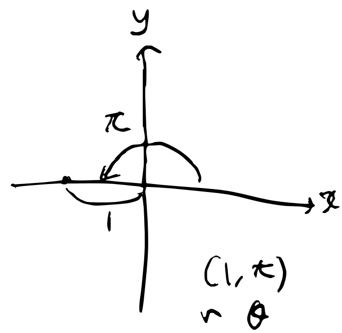


in polar coordinates is





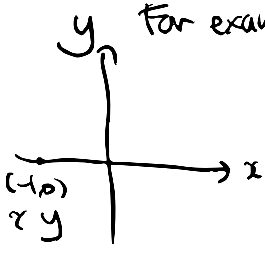
in polar coordinates is



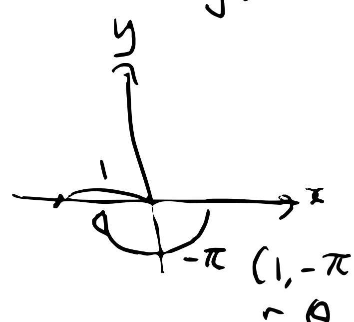
Note: * r is always $r \geq 0$, because r is the distance.

* θ is usually taken in between 0 and 2π, but the difference by 2π does not make any difference.

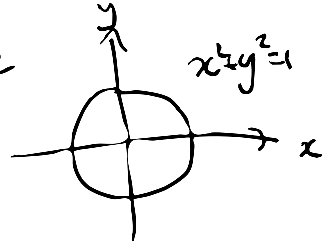
For example,



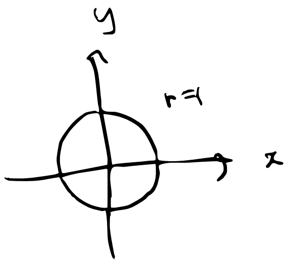
can also be expressed in polar coordinates as



(- because we rotate clockwise, not counter-clockwise).

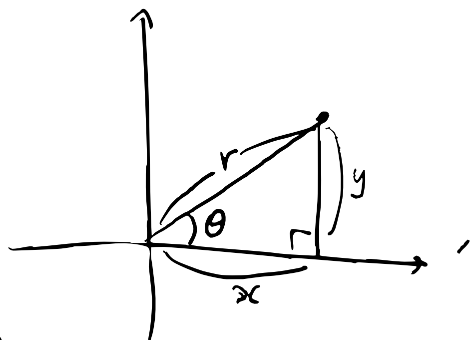
The graph of the circle  $x^2+y^2=1$ in rectangular coords

becomes



in polar coordinates, which is nice.

Given the diagram



we have the relations

$$r = \sqrt{x^2+y^2}$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$\tan\theta = \frac{y}{x}$$

Using these, you can convert between rectangular coordinates (in 2D) and polar coordinates.

polar → rectangular:

$(r,\theta) \rightsquigarrow$ $x = r\cos\theta$, $y = r\sin\theta$

rectangular → polar:

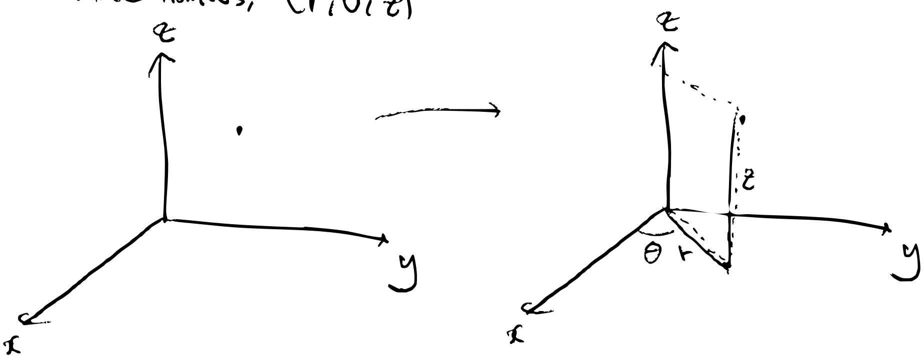
$(x,y) \rightsquigarrow$ $r = \sqrt{x^2+y^2}$

θ is such that $\tan\theta = \frac{y}{x}$ (fancier: $\theta = \tan^{-1}\left(\frac{y}{x}\right)$)

Cylindrical coordinates (3D)

This is very much related to polar coordinates. In fact, cylindrical coordinates are just polar coordinates with heights

We use three numbers, (r, θ, z)



Similar to polar coordinates, cylindrical coordinates have the following relations with rectangular coordinates (in 3D).

- $r = \sqrt{x^2+y^2}$

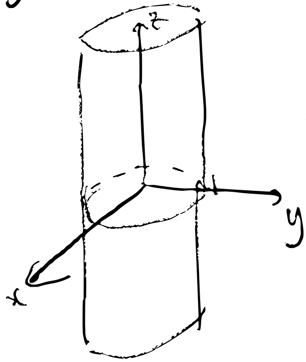
$x = r\cos\theta$

$y = r\sin\theta$

$\tan\theta = \frac{y}{x}$

$z = z$

Cylindrical coordinates are great at expressing cylinders.



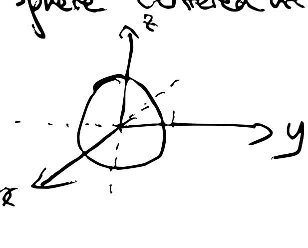
$r=1$ represents a cylinder of radius 1 that extends indefinitely along the z-axis.

Exercises

- Find the rectangular coordinates (in 2D) of the point $(r,\theta)=(2,\frac{\pi}{4})$ in polar coordinates.
- Find the polar coordinates of the point $(x,y)=(1,-1)$ in rectangular coordinates (in 2D).
- Find the rectangular coordinates (in 3D) of the point $(r,\theta,z)=(4,\frac{4\pi}{3},1)$ in cylindrical coordinates.
- Find the cylindrical coordinates of the point $(x,y,z)=(0,-1,3)$ in rectangular coordinates (in 3D).

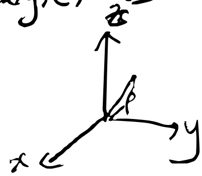
Spherical coordinates (3D)

Recall that the equation of the sphere centered at the origin with radius 1 is $x^2+y^2+z^2=1$.



This is because, in 3D rectangular coordinates, the distance between the origin and the point (x,y,z) is

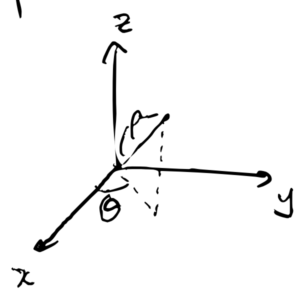
$$\rho=\sqrt{x^2+y^2+z^2}.$$



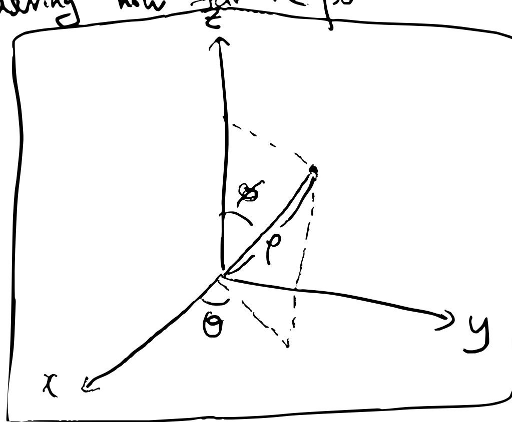
<u>Note</u> This is different from $r=\sqrt{x^2+y^2}$, which is the distance in 2D only.

Spherical coordinates use this 3D distance ρ, with <u>two different angles θ and ϕ</u>, to tell the location of a given point.

- θ is the same θ as is in cylindrical coordinates; namely, how far the point is from the x-axis.



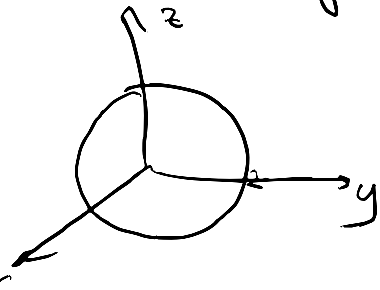
- ϕ is a new angle measuring how far the point is from the z-axis.



Spherical coordinates can express many shapes in simple equations.

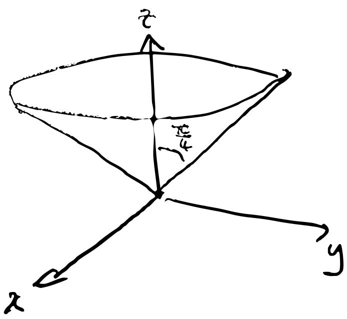
Examples

- A sphere centered at the origin with radius 2.



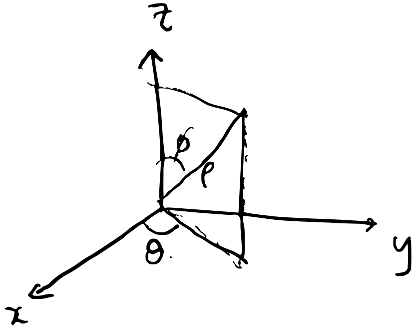
$\rho = 2$.

- A cone centered at the origin, facing upwards, with the surface tilted $\frac{\pi}{4}$ radians away from the z-axis.

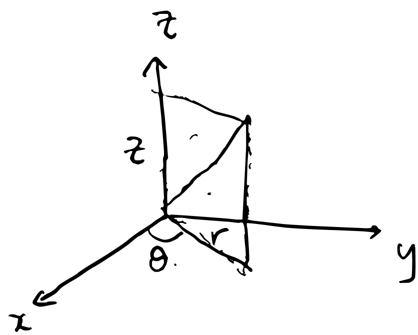


$\phi = \frac{\pi}{4}$

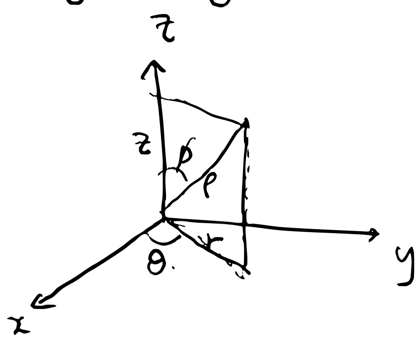
How does one calculate with spherical coordinates? Recall the diagram for spherical coordinates,



This has many things in common with the corresponding diagram for cylindrical coordinates.



Merging these together, we get



So, $z = \rho\cos\phi$, $r = \rho\sin\phi$. Together with $\theta = \theta$, we get a complete set of relations between spherical and cylindrical coordinates

$r = \rho \sin\phi$
$\theta = \theta$
$z = \rho \cos\phi$

Spherical $\longrightarrow$ Cylindrical

$(\rho, \theta, \phi) \rightsquigarrow (r, \theta, z) = (\rho \sin\phi, \theta, \rho\cos\phi)$

Cylindrical $\longrightarrow$ Spherical

$(r, \theta, z) \rightsquigarrow (\rho, \theta, \phi) = \left(\sqrt{r^2+z^2}, \theta, \tan^{-1}\left(\frac{r}{z}\right)\right)$

<u>Exercise</u> Explain why $\rho = \sqrt{r^2+z^2}$.

Since we know how to convert back & forth between cylindrical coordinates and rectangular coordinates (in 3D) we in turn get to know how to convert back and forth between spherical coordinates and rectangular coordinates (in 3D).

Spherical $\longrightarrow$ Rectangular (in 3D)

$(\rho, \theta, \phi) \rightsquigarrow (x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi)$

Rectangular (in 3D) $\longrightarrow$ Spherical

$(x, y, z) \rightsquigarrow$ $\rho = \sqrt{x^2+y^2+z^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

Exercises

- Convert the point $(x, y, z) = (0, -2, 0)$ in rectangular coordinates (in 3D) to spherical coordinates.
- Identify the surface whose equation in spherical coordinates is $\rho\cos(\phi) = 3$.
- Convert the point $(\rho, \theta, \phi) = (2, \pi/4, \pi/4)$ to rectangular coordinates (in 3D).
- Express the equation $\phi = \frac{\pi}{4}$ for the cone in terms of cylindrical coordinates.

Note Similarly with polar/cylindrical coordinates, there are constraints for the values of ρ, θ, ϕ for spherical coordinates.

- $\rho \geq 0$, because it's a distance.
- θ, as in polar/cylindrical coordinates, is taken typically in between 0 and 2π, but adding/subtracting a multiple of 2π would not affect at all.
- ϕ is taken in between 0 and π. $\phi = 0$ is the positive z-axis, and $\phi = \pi$ is the negative z-axis.